B) Theory of Groups (231103)

Time : Two Hours **Max. Marks : 60**

Instructions to Candidates :
1. Do not write anything on question paper except Seat No.
2. Graph or diagram should be drawn with the black ink pen being used for writing paper or black HB pencil.
3. Students should note, no supplement will be provided.
4. All questions are compulsory.
5. Figures to right indicate full marks.

1. a) Attempt **any six** of the following. **6**

i) The order of the identity element is any group is always -----

a) 0 b) 1
c) -1 d) 2

ii) If G is abelian group w.r.t. multiplication such that $b^{-1}a^{-1}ab = ...$, $\forall a, b \in G$.

a) a b) b
c) e d) ab

iii) If A and B are two subgroups of a finite group G whose orders are relatively prime then $A \cap B = ...$

a) $\{e\}$ b) A
c) B d) None of these

iv) H and K be subgroups of a group G then $H \cup K$ is subgroup of G iff -----

a) $H = K$ b) $HK = e$
c) either $H \subseteq K$ or $K \subseteq H$ d) None

v) $f : G \to G'$ be a group homomorphism then f is one - one iff kerf = ---

a) $\{e\}$ b) G
c) G' d) None of these

vi) An isomorphism is an automorphism from group G onto G' if -----

a) $G = G'$ b) $G \neq G'$
c) $G' \subseteq G$ d) None of these

vii) A (m, n) encoding function e can detect K or fewer errors iff minimum distance of e is at least -----

a) K b) $K-1$
c) $K+1$ d) $K-2$

viii) The Hamming distance between x = 1011 and y = 1100 is -----

a) 2 b) 3
c) 1 d) 4

b) Attempt **any six** of the following. 6

i) Define finite abelian group.

ii) In group $\langle Z, +\rangle$, find 2^{-4}

iii) Define cyclic group.

iv) State Lagrange's theorem.

v) Let $f: G \rightarrow G'$ is a group homomorphism from G into group G' such that e is identity of G than write down identity of G'.

vi) Define homomorphism of groups.

vii) Define weight of the word.

viii) The minimum distance of (2, 4) encoding function e is 2. How many errors will e defect?

2. Attempt **any six** of the following. 12

i) Let $G = \{0, 1, 2\}$ and define $*$ on G by $a * b = |a - b|$, prepare composition table for $*$.

ii) If G is a group, then prove that $\left(a^{-1}\right)^{-1} = a \ \forall a \in G$.

iii) Define Euler's totient function and find $\phi(7)$.

iv) Let $G = \{1, -1, i, -i\}$ be group under multiplication and $H = \{1, -1\}$ be its subgroup, find all right cosets of H in G.

v) Define subgroup of a group. Show that the set of odd integers is not a subgroup of group $\langle Z, +\rangle$.

vi) Let $\langle G, *\rangle$ and $\langle G', *'\rangle$ be groups with identity elements e and e' respectively. Show that $f : G \to G'$ defined by $f(a) = e' \ \forall\, a \in G$ is homomorphism.

vii) Let $f : G \to G'$ be a group homomorphism. if e is the identity of G then prove that f(e) is the identity of G'.

viii) Prove that $\delta(x, y) = \delta(y, x)$ if $x, y \in B^m$.

ix) Compute $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

3. Attempt **any four** of the following. **12**

i) Show that set $G = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$ is a group under usual multiplication of matrices.

ii) If G be a group and $a, b, c \in G$ then prove that
   i) $ab = ac \Rightarrow b = c$
   ii) $ba = ca \Rightarrow b = c$

iii) Prove that every group of prime order is cyclic.

iv) Find remainder when 15^{27} is divided by 8.

v) Let $\langle Z, +\rangle$ be additive group of integers and $G = \{1, -1, i, -i\}$ be multiplicative group. Show that $f : Z \to G$ defined by $f(n) = i^n \ \forall\, n \in Z$ is a group homomorphism. Find its kernel.

vi) Find minimum distance of (2, 4) encoding function e given below : . $e(00) = 0000$, $e(10) = 0110$, $e(01) = 1011$, $e(11) = 1100$ How many errors will e detect?

4. Attempt **any three** of the following. **12**

i) Show that group G is abelian if and only if $(ab)^2 = a^2b^2 \ \forall\, a, b \in G$

ii) Let Q^+ be set of all positive rationals. For a $b \in Q^+$, define $a * b = \frac{ab}{7}$ show that $\langle Q^+, * \rangle$ is an abelian group.

iii) Let H be a subgroup of a group G and for $g \in G$ define gH_g^{-1} by $gH_g^{-1} = \{gh_g^{-1} / h \in H\}$. Show that gH_g^{-1} is subgroup of G.

iv) Let $\langle IR, + \rangle$ be a group of reals and $\langle IR^+, \cdot \rangle$ is a group of positive reals. Define $f : IR^+ \to IR$ by $f(x) = \int_1^x \frac{1}{t} dt, \forall x \in IR^+$. Show that f is an isomorphism.

v) Let e be an (2, 5) encoding function defined by $e(00) = 00000$, $e(01) = 01110$, $e(10) = 10101$, $e(11) = 11011$ show that e is a group code.

5. Attempt **any two** of the following. 12

i) Prove that nonempty subset H of a group G is a subgroup of G if and only if $\forall\ a, b \in H \Rightarrow ab^{-1} \in H$.

ii) Let $f : G \to G'$ be a group homomorphism and $0(a)$ is finite then prove that f is one - one if and only if $0(f(a)) = 0(a)\ \forall\ a \in G$.

iii) Determine the coset leaders for $N = e_H(B^m)$ for given parity check matrix $H = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ and also compute syndrome for each coset leader. Decode word 0101 relative to maximum likelihood decoding function.

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