B) Theory of Groups (231103)

Time : Two Hours **Max. Marks : 60**

Instructions to Candidates :

1. Do not write anything on question paper except Seat No.
2. Graph or diagram should be drawn with the black ink pen being used for writing paper or black HB pencil.
3. Students should note, no supplement will be provided.
4. All questions are compulsory.
5. Figures to the right indicate full marks.

1. a) Attempt **any six** of the following. 6

i) The inverse of $\bar{7}$ in the group < G x 8>; where $G = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$, is -------

I) $\bar{1}$ II) $\bar{3}$

III) $\bar{5}$ IV) $\bar{7}$

ii) If the order of an element a in the group G is 1 then ------

I) $a = 0$ II) $a = a^{-1}$

III) $a = a^2$ IV) $a = e$

iii) A group <G +> is abelian if $\forall a, b \in G$, -----

I) $ab = ba$ II) $a * b = b * a$

III) $a \oplus b = b \oplus a$ IV) $a + b = b + a$

iv) The group <Z +> is cyclic group generated by -------

I) 0 II) 1

III) -1 IV) 2

v) If H is subgroup of finite group then -------

I) O(G) / O(H) II) O(H) / O(G)

III) O(H) X O (G) IV) None of these

vi) A homomorphism $f : G \rightarrow G'$ is an isomorphism if f is -------

I) One-one II) Onto

III) One-one and onto IV) None

vii) The homomorphic image of an cyclic group is -----
I) Cyclic
II) Non abelian
III) Not cyclic
IV) None

viii) An (m.n) encoding function $e : B^m \rightarrow B^n$ can detect K or fewer errors if and only if its minimum distance is ----------
I) $\geq K + 1$
II) $\leq K + 1$
III) $= K + 1$
IV) $= K$

b) Attempt **any six** of the following. 6

i) Define : inverse of an element in a group $< G, * >$

ii) In the group $< z_6, +_6 >$ find $(\bar{2})^4$

iii) What is meant by subgroup of a group.

iv) Let G be a group of order 7 Is G cyclic? Why?

v) Explain: The kernel of a homomorphism.

vi) Define: Encoding function

vii) If $f : G \rightarrow G'$ be an isomorphism from group G to group G' and $a \in G$, then write down value of O[f(a)].

viii) The minimum distance of (3,9) encoding function e is 3. How many errors will e detect?

2. Attempt **any six** of the following. 12

a) Show that the set {1,2,3} under multiplication modulo -4 is not group.

b) Define : Euler's totient function $\phi(n)$ and hence find $\phi(\in)$

c) Find all left cosets of subgroup H in group G if $H = \{\bar{0}, \bar{4}\}$ and $G = \left\langle z_8^1, +_8 \right\rangle$

d) Prove that every cyclic group is abelian.

e) Let <IR, +> be a group of reals under usual addition show that $f: IR \to IR$ defined by $f(x) = 2x \ \forall x \in IR$ is a group homomorphism.

f) If $f: G \to G'$ is a group homomorphism; then prove that $f(a^{-1}) = [f(a)]^{-1} \ \forall a \in G$.

g) For $x, y \in B^m$, Prove that $\delta(x, y) = 0$ if and only if $x = y$.

h) Compute $\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

i) Find weight of $x = 1101101 \in B^7$ and $y = 111111100 \in B^9$

3. Attempt **any four** of the following. 12

a) Show that $G = \{1, -1, i, -i\}$ where $i = \sqrt{-1}$ is a group under usual multiplication of complex numbers.

b) Let G be a group and $a, b \in G$. Then show that the equation $ax = b$ have unique solution in G

c) If A and B are subgroups of a finite group G whose orders are relatively prime then show that $A \cap B = \{e\}$

d) Find the remainder obtained when 3^{54} is divided by 11.

e) Let <IR, +> and <IR* .> be groups where $IR^* = IR - \{o\}$. Show that a mapping $f: IR \to IR^*$ defined by $f(x) = e^x \ \forall x \in IR$ is a group homomorphism. Find kernel of f.

f) Find the minimum distance of the (2,5) encoding function e; given below : e(00) = 00000, e(10) = 10110, e(01) = 01011 e(11) = 11101. How many errors e can defect?

4. Attempt **any three** of the following. 12

a) Define: order of an element in a group. If G be a group and $a \in G$ such that $0(a) = 2$, then prove that G is abelian.

b) Show that $G = \left\{ \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \Big/ \alpha \in IR \right\}$

is an abelian group under matrix multiplication.

c) Let G be a group and $a \in G$. Denote N(a) by $N(a) = \{x \in G \mid xa = ax\}$. Show that N(a) is subgroup of group G.

d) Let G be a group; and $f : G \to G$ be a mapping defined by $f(x) = x^{-1}\ \forall x \in G$. Prove that G is abelian if and only if f is an isomorphism.

e) Show that the (2,5) encoding function $e : B^2 \to B^5$ defined by e(00) = 00000, e(01) = 01110, e(10) = 10101, e(11) = 11011 is a group code.

5. Attempt **any two** of the following. **12**

a) If H and K are two subgroups of a group G then prove that $H \cap K$ is a subgroup of G. Is $H \cup K$, a subgroup of G? Justify.

b) Prove that homomorphic image of an abelian group is abelian. Is the converse true? Justify your answer.

c) Let

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

be a parity check matrix determine coset leaders for $N = e_H(B^m)$ for the given parity check matrix H and also compute the syndrome for each coset Leader. Decode the word 001110 relative to a maximum likelihood decoding function.

****